International Journal of Theoretical Physics, Vol. 35, No. 10, 1996

Clifford Algebra Structure and Einstein's Equations

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Received December 7, 1995

We point out that a Clifford algebra representation for the Riemannian curvature leads to an equation for gravity similar to the Yang–Mills equation, in a gauge model for gravity with the Lorentz group. Einstein's equation of general relativity emerges as a natural solution in this approach.

Dirac-like equations are the basis of the Bargmann–Wigner classification for an irreducible representation of the Poincaré group. Since both the photon and the neutrino fit into the Bargmann–Wigner scheme (Bargmann and Wigner, 1948), they are associated with zero-mass representations of the Poincaré group. This is due to the fact that the dynamical Maxwell equations can be written in a form similar to Weyl's equation (Laporte and Uhlenbeck, 1931) for the neutrino. Moreover, the linearized Einstein equation can be similarly formulated in such a form (Doria, 1973). In the present paper we point out that if we include a Yang–Mills (YM) formulation for gravity with the Lorentz group in a structure that generalizes the Clifford algebra, then we are led naturally to Einstein's equations of general relativity (GR).

The mathematical setting we consider is given by the following assumptions: let $\mathcal{M}$ be a smooth, four-dimensional space-time manifold endowed with a nondegenerate metric tensor $\mathbf{g}$ with signature -2 and let $P(\mathcal{M}, G)$ be a G-principal bundle over $\mathcal{M}$, where G is the Poincaré group; let $\otimes T\mathcal{M}$ be $\mathcal{M}$'s tensor bundle and $\Omega\mathcal{M}$ its covariant exterior bundle endowed with the differential operators d and δ. Let us consider also associated vector bundles $E(\mathcal{M}, G)$, where E is a representation vector space for G, and let $C\mathcal{M}$ be the Clifford bundle with respect to $\mathcal{M}$'s metric tensor, isomorphic to $\Omega\mathcal{M}$. Here the objects are constructed on the bundles $E \otimes C\mathcal{M}$ and $E \otimes \Omega\mathcal{M}$. We also suppose the canonical isomorphism ι: $C\mathcal{M} \cong \Omega\mathcal{M}$, which is extended to ι:

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0020-7748/96/1000-2117$09.50/0

$E \otimes \Omega\mathcal{M}$, and which is locally given by the assumption that the cotangent bundle $T\mathcal{M} \subset \Omega\mathcal{M}$ and the generators of $C\mathcal{M}$ are covectors $\iota^{-1}T\mathcal{M}$. Finally, we consider that in a local system, $T\mathcal{M}$ is spanned by $\{dx^\mu\}$, $\mu = 0, 1, 2, 3$, in such a way that we can identify $\iota\gamma^\mu = dx^\mu$, where γ^μ are Dirac operators. In this structure we have an associative algebra with identity over a ring of C^5 real-valued functions defined locally on $\mathcal{M}$.

We recall that the $\mathcal{G}$-algebra of the Poincaré group is a vector space, given by the direct sum $\mathcal{G} = \mathcal{R} \oplus \mathcal{T}$, where $\mathcal{R}$ and $\mathcal{T}$ are, respectively, the rotational (Lorentz) and translational sectors of $\mathcal{G}$. A connection 1-form $\Gamma = J^m_n \Gamma^n_{m\mu} dx^\mu$ is an $\mathcal{R}$-valued connection in the Lorentz sector, where J^m_n are generators of the Lorentz group which define a basis in the $\mathcal{R}$-vector space. Here Latin indices $m, n, \ldots$ are group algebra indices. The curvature of Γ is a 2-form also $\mathcal{R}$-valued and given by $F = d\Gamma + \Gamma \wedge \Gamma$. From the gauge point of view Γ is a potential of a non-Abelian gauge theory and F is the corresponding gauge field. This curvature satisfies the Bianchi identity $dF + [\Gamma, F] = 0$.

In order to analyze gravity in the gauge context, we follow the orthodox point of view of classical gauge theories. We recall that the assumption of a metric tensor $\mathbf{g}$ on the space-time manifold $\mathcal{M}$ brings an important consequence. Indeed, there is only one torsionless connection Γ preserving $g_{\mu\nu}$, i.e., it is such that the covariant derivative $\nabla^\mu g_{\mu\nu}$ is vanishing if calculated with respect to the connection Γ. Such a very special metric connection is the Levi-Civitá connection, whose curvature is given by the Riemann tensor. This fact points out a link of GR to space-time which is much stronger than that of the usual gauge theories. It is responsible for the fundamental difference between gravitation and gauge theories, since gravitation is more intimately related to the background space-time manifold.

In this framework the YM equation is written as a breaking of the dual symmetry of the Bianchi identity for the Lorentz group. Sources to be inserted in the YM equation can be in general Noether current densities, whose "charges" are the generators of the Lorentz group:

$$\delta F + *^{-1}[\Gamma, *F] = *\Lambda \tag{1}$$

Since locally the space-time $\mathcal{M}$ is endowed with a Riemannian structure when we consider Γ as a Levi-Civitá connection, we thus obtain for the YM equation in components

$$\partial^\lambda \tilde{F}^m_{n\mu\lambda} + \Gamma^m_p{}^\lambda \tilde{F}^p_{n\mu\lambda} - \Gamma^p_n{}^\lambda \tilde{F}^m_{p\mu\lambda} = \Lambda^m_{n\mu} \tag{2}$$

where the tilde stands for the dual. In a dual basis (where the tetrads are $h^m_\mu = \delta^\mu_\nu$) the above equation becomes in a Riemannian manifold (Stedile and Duarte, 1995)

$$\tilde{\nabla}^{\lambda}\tilde{R}^{\alpha}_{\beta\mu\lambda} = \Lambda^{\alpha}_{\beta\mu} \tag{3}$$

where we have assumed the transformation $\tilde{R}^{\alpha}_{\beta\mu\lambda} = h^{\alpha}_{m}\tilde{R}^{m}_{n\mu\lambda}h^{n}_{\beta}$, which represents an isomorphism between the tangent space and space-time given in terms of the tetrad fields. In the above equation the operator $\tilde{\nabla}$ denotes the covariant derivative with respect to the first pair of indices (α, β).

For the Clifford algebra we consider eight generators $\{\kappa^{\mu}_{i}\}$, which satisfy the product rules

$$[\kappa^{\mu}_{i}, \kappa^{\nu}_{j}]_{-} = 0, \qquad [\kappa^{\mu}_{i}, \kappa^{\nu}_{i}]_{+} = 2g^{\mu\nu} \qquad (i, j = 1, 2) \tag{4}$$

The generators κ^{μ}_{i} define a generalized space-time algebra $\mathscr{C}$ whose derivative operator D_{μ} is linear and acts on the objects of $\mathscr{C}$. Hence, if f is a scalar function and if $\mathbf{a}$ and $\mathbf{b}$ denote generic elements of $\mathscr{C}$, then we have the following properties (Doria, 1975):

$$D_{\mu}f = \partial_{\mu}f, \qquad D_{\mu}\kappa^{\nu}_{i} = -\Gamma^{\nu}_{\mu\rho}\kappa^{\rho}_{i}, \qquad D_{\mu}(\mathbf{ab}) = (D_{\mu}\mathbf{a})\mathbf{b} + \mathbf{a}(D_{\mu}\mathbf{b}) \tag{5}$$

where $\partial_{\mu} = \partial/\partial x^{\mu}$. If we define now $\kappa^{\mu\nu}_{i} = \frac{1}{2}[\kappa^{\mu}_{i}, \kappa^{\nu}_{i}]_{-}$, we obtain easily

$$D_{\alpha}\kappa^{\mu\nu}_{i} = -\Gamma^{\mu}_{\alpha\beta}\kappa^{\beta\nu}_{i} - \Gamma^{\nu}_{\alpha\beta}\kappa^{\mu\beta}_{i} \tag{6}$$

Although there are some restrictions when we try to construct a global system of Clifford bases on arbitrary space-times (Geroch, 1968), we can endow both the tangent and the cotangent spaces to $\mathcal{M}$ with a Clifford structure. If we choose a representation for the κ-generators according to $\kappa^{\mu}_{1} = \gamma^{\mu} \otimes \mathbf{1}$ and $\kappa^{\mu}_{2} = \mathbf{1} \otimes \gamma^{\mu}$, where $\mathbf{1}$ is the identity matrix, then a local cotangent basis $\{\gamma^{\mu}\}$ at a point $p \in \mathcal{M}$ has the property $[\gamma^{\mu}, \gamma^{\nu}]_{+} = 2g^{\mu\nu}$. Defining now the quantities $S_1 = 2\Lambda_{\rho\mu\nu}\kappa^{\rho}_{1}\kappa^{\mu\nu}_{2}$, $S_2 = 2\Lambda_{\mu\nu\rho}\kappa^{\mu\nu}_{1}\kappa^{\rho}_{2}$, and $V = \frac{1}{2}V_{\mu\nu\rho\sigma}(\kappa^{\mu\nu}_{1}\kappa^{\rho\sigma}_{2} + \kappa^{\mu\nu}_{2}\kappa^{\rho\sigma}_{1})$, where $V_{\mu\nu\rho\sigma}$ are arbitrary functions of both $\Gamma^{\alpha}_{\beta\mu}$ and $\partial_{\mu}\Gamma^{\alpha}_{\beta\nu}$, we obtain that $\kappa^{\mu}_{i}D_{\mu}V = S_i$. These assumptions imply that

$$V_{\mu\nu\rho\sigma;\alpha} + V_{\mu\nu\alpha\rho;\sigma} + V_{\mu\nu\sigma\alpha;\rho} = 0, \qquad V^{\mu}_{\nu\rho\sigma;\mu} = \Lambda_{\nu\rho\sigma} \tag{7}$$

where semicolon denotes the covariant derivative with respect to the connection Γ, relative to the first pair of indices.

The functional dependence of $V_{\mu\nu\rho\sigma}$ on the Levi-Civitá connection and on its derivative states that those $V_{\mu\nu\rho\sigma}$ can be in particular the components of the dual Riemann curvature tensor, and in consequence the first expression of equations (7) leads to the Bianchi identity, since $\tilde{R}^{\alpha\beta\lambda\sigma} = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}R_{\gamma\delta\tau\rho}\epsilon^{\tau\rho\lambda\sigma}$. In this case the second expression of (7) is the same YM equation given in (3). Otherwise, if we choose

$$\Lambda_{\nu\rho\sigma} = (T_{\nu\sigma;\rho} - \tfrac{1}{2}g_{\nu\sigma}\partial_{\rho}T) - (T_{\nu\rho;\sigma} - \tfrac{1}{2}g_{\nu\rho}\partial_{\sigma}T)$$

where **T** is the energy-momentum tensor of a source of gravity, then both equations (7) are satisfied if and only if

$$R_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T \tag{8}$$

which is Einstein's equation of GR.

Hence, the framework presented here shows a link between a YM gauge formulation for gravity with the Lorentz group, and a Clifford algebra representation. In consequence, Einstein's equations emerge naturally when we assume a Levi-Civitá connection.

ACKNOWLEDGMENT

This work was supported in part by CNPq (a Brazilian government agency).

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